If f is concave, then these conditions are also sufficient:

PROPOSITION C.7 *Suppose $f \in C^1$ is a concave function and X is a convex set. Then if a point $\mathbf{x}^*$ satisfies*

$$\nabla \mathbf{f}(\mathbf{x}^*)^\top (\mathbf{x} - \mathbf{x}^*) \leq 0, \ \forall \mathbf{x} \in X,$$

it is a global maximum.

In the nonconvex unconstrained case, local optimality is guaranteed by the following *second-order sufficiency conditions*:

PROPOSITION C.8 *If $f \in C^2$ and $X = \Re^n$, then if a point $\mathbf{x}^*$ satisfies*
(i)$\nabla \mathbf{f}(\mathbf{x}^*) = 0$
(ii)$\nabla^2 \mathbf{f}(\mathbf{x}^*)$ *is positive definite,*
it is a local maximum.

There are no general sufficient conditions for global optima in the nonconvex case.

Equality and Inequality Constraints

Suppose the set X is defined by a set of linear equalities. That is, $X = \{\mathbf{x} : h(\mathbf{x}) = \mathbf{b}\}$, where $h : \Re^n \to \Re^m$ (i.e., $\mathbf{h}(\mathbf{x}) = (h_1(\mathbf{x}), \ldots, h_m(\mathbf{x}))$) so the optimization problem to solve is

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{h}(\mathbf{x}) = \mathbf{b}. \end{aligned}$$

We require the following definition:

DEFINITION C.9 *A point $\mathbf{x}^*$ satisfying $\mathbf{h}(\mathbf{x}^*) = \mathbf{b}$ is said to be a* **regular point** *of the constraints $\mathbf{h}(\mathbf{x}) = \mathbf{b}$ if the vectors $\nabla \mathbf{h}_1(\mathbf{x}), \ldots, \nabla \mathbf{h}_m(\mathbf{x})$ are linearly independent.*

The assumption of regularity of x^* is an example of what is called a *constraint qualification,* a condition that ensures that the first-order conditions correctly identify a local optimum.

We then have the following first-order necessary conditions:

PROPOSITION C.9 *Suppose $f \in C^1$ and $\mathbf{x}^*$ is a local maximum of the function f over the constraint set $X = \{x : \mathbf{h}(\mathbf{x}) = \mathbf{b}\}$. Then if $\mathbf{x}^*$ is a regular point, there exist a vector $\boldsymbol{\pi} \in \Re^m$ such that*

$$\nabla \mathbf{f}(\mathbf{x}^*) - \boldsymbol{\pi}^\top \nabla \mathbf{h}(\mathbf{x}^*) = 0.$$

A vector $\boldsymbol{\pi}$ above is called a *Lagrange multiplier* of the constraints $\mathbf{h}(\mathbf{x}) = \mathbf{b}$.

If the constraint set is defined by inequalities, so the problem is

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{x}) \leq \mathbf{d}, \end{aligned}$$

where $\mathbf{g} : \Re^n \to \Re^m$, then similar conditions apply. Indeed, the definition of a regular point in this case is: